

# Applications of the Canonical Ramsey Theorem to Geometry

William Gasarch <sup>\*</sup>

Univ. of MD at College Park

Sam Zbarsky <sup>†</sup>

Montgomery Blair High School

February 22, 2013

## Abstract

Let  $\{p_1, \dots, p_n\} \subseteq \mathbb{R}^d$ . We think of  $d \leq n$ . How big is the largest subset  $X$  of points such that all of the distances determined by elements of  $\binom{X}{2}$  are different? We show that  $X$  is at least  $\Omega((n^{1/(6d)}(\log n)^{1/3})/d^{1/3})$ . This is not the best known; however the technique is new.

Assume that no three of the original points are collinear. How big is the largest subset  $X$  of points such that all of the areas determined by elements of  $\binom{X}{3}$  are different? We show that, if  $d = 2$  then  $X$  is at least  $\Omega((\log \log n)^{1/186})$ , and if  $d = 3$  then  $X$  is at least  $\Omega((\log \log n)^{1/396})$ . We also obtain results for countable sets of points in  $\mathbb{R}^d$ .

All of our results use variants of the canonical Ramsey theorem and some geometric lemmas.

## 1 Introduction

Let  $\{p_1, \dots, p_n\} \subseteq \mathbb{R}^d$ . We think of  $d \leq n$ . How big is the largest subset  $X$  of points such that all of the distances determined by elements of  $\binom{X}{2}$  are different? Assume that no three of the original points are collinear. How big is the largest subset  $X$  of points such that all of the areas determined by elements of  $\binom{X}{3}$  are different?

**Def 1.1** Let  $a \geq 1$ . Let  $h_{a,d}(n)$  be the largest integer so that if  $p_1, \dots, p_n$  are any set of  $n$  distinct points in  $\mathbb{R}^d$ , no  $a$  points in the same  $(a-2)$ -dimensional space, then there exists a subset  $X$  of  $h_{a,d}(n)$  points for which all of the volumes determined by elements of  $\binom{X}{a}$  are different. The  $h_{a,d}(n)$  *problem* is the problem of establishing upper and lower bounds on  $h_{a,d}(n)$ . The definition extends to letting  $n$  be an infinite cardinal  $\alpha$  where  $\aleph_0 \leq \alpha \leq 2^{\aleph_0}$ .

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<sup>\*</sup>University of Maryland at College Park, Department of Computer Science, College Park, MD 20742.  
gasarch@cs.umd.edu

<sup>†</sup>Montgomery Blair High School, Silver Spring, MD, 20901 sa\_zbarsky@yahoo.com

Below we summarize all that is know about  $h_{a,d}(n)$  (to our knowledge).

1. Erdős [7], in 1946, showed that the number of distinct differences in the  $\sqrt{n} \times \sqrt{n}$  grid is  $\leq O(\frac{n}{\sqrt{\log n}})$ . Therefore  $h_{2,2}(n) \leq O\left(\sqrt{\frac{n}{\log n}}\right)$ . For  $a \geq 3$  We do not know of any nontrivial upper bounds on  $h_{2,d}$ . (The set  $n^{1/d} \times \dots \times n^{1/d}$  has many points collinear and hence cannot be used to obtain an upper bound.)
2. Erdős [8], in 1950, showed that, for  $\aleph_0 \leq \alpha \leq 2^{\aleph_0}$ ,  $h_{2,d}(\alpha) = \alpha$ .
3. Erdős considered the  $h_{2,d}(n)$  problem 1957 [9] and 1970 [10]. In the latter paper he notes that  $h_{2,2}(7) = 3$  [12] and  $h_{2,3}(9) = 3$  [5]. Erdős conjectured that  $h_{2,1}(n) = (1+o(n))n^{1/2}$  and notes that  $h_{2,1}(n) \leq (1+o(n))n^{1/2}$  [15].
4. Komlos, Sulyok and Szemerédi [21], in 1975, show that  $h_{2,1}(n) \geq \Omega(\sqrt{n})$  though they state it in different terms.
5. Erdős [11] considered the  $h_{2,d}(n)$  problem in 1986. He states *It is easy to see that  $h_{2,d}(n) > n^{\epsilon_d}$  but the best possible value of  $\epsilon_d$  is not known.  $\epsilon_1 = \frac{1}{2}$  follows from a result of Ajtai, Komlos, Sulyok and Szemerédi [21].* (We do not know why he added Ajtai who was not an author on that paper.)
6. Avis, Erdős, and Pach [2], in 1991, showed that for all sets of  $n$  points in the plane, for almost all  $k$ -subsets  $X$  where  $k = o(n^{1/7})$ , the elements of  $\binom{X}{2}$  determine different distances. Hence, for example,  $h_{2,2}(n) = \Omega(n^{1/7+\epsilon})$ .
7. Thiele [25], in his PhD thesis from 1995, has as Theorem 4.33, that for all  $d \geq 2$ ,  $h_{2,d} = \Omega(n^{1/(3d-2)})$ .
8. Charalambides [3], in 2012, showed that  $h_{2,2}(n) = \Omega(n^{1/3}/\log n)$ .
9. We know of no references to  $h_{a,d}$  for  $a \geq 3$  in the literature.
10. We believe that this is the first paper to define  $h_{a,d}$  in its full generality.

**Note 1.2** The problem of  $h_{2,2}$  is similar to but distinct from the *Erdős Distance Problem*: give a set of  $n$  points in the plane how many distinct distances are guaranteed. For more on this problem see [17, 18]. The problem of  $h_{3,2}$  is similar to but distinct from the problem of determining, given  $n$  points in the plane no three collinear, how many distinct triangle-areas are obtained (see [6] and references therein). We do not know of any reference to a higher dimensional analog of these problems.

Below we list our result. For two of our results stronger results are known and in the above list; however, our proofs are very different. We find our proofs simpler.

- $h_{2,d}(n) \geq \Omega((n^{1/(6d)}(\log n)^{1/3})/d^{1/3})$ . (Torsten has a better result.)

- $h_{3,2}(n) \geq \Omega((\log \log n)^{1/186})$ .
- $h_{3,3}(n) \geq \Omega((\log \log n)^{1/396})$ .
- $h_{2,d}(\aleph_0) = \aleph_0$ . (Erdős had a more general result.)
- $h_{3,2}(\aleph_0) = \aleph_0$ .
- $h_{3,3}(\aleph_0) = \aleph_0$ .

Our proofs have two ingredients: (1) upper bounds on variants of the canonical Ramsey numbers, and (2) geometric lemmas about points in  $\mathbb{R}^d$ .

In Section 2, 3, and 4 we define terms, prove lemmas, and finally prove an upper bound on a variant of the canonical Ramsey Theorem. Our proof uses some ideas from the upper bound on the standard canonical Ramsey number,  $ER(k)$ , due to Lefmann and Rödl [23]. In Section 5 we prove a geometric lemma about points in  $\mathbb{R}^d$ . In Section 6 we use our upper bound and our geometric lemma to prove lower bounds on  $h_{2,d}(n)$ . In Section 7 we prove the needed variant of the canonical Ramsey theorem, and the needed geometric lemmas, to obtain lower bounds on  $h_{3,2}(n)$  and  $h_{3,3}(n)$ . In Section 8 we use known theorems and our geometric lemmas to obtain results about countable sets of points. In Section 9 we speculate about lower bounds for  $h_{a,d}$  for  $a \geq 3$ . In Section 10 we list open problems.

## 2 Variants of the Canonical Ramsey Theorem

**Notation 2.1** Let  $n \in \mathbb{N}$ .

1.  $[n]$  is the set  $\{1, \dots, n\}$ .
2. If  $X$  is a set and  $0 \leq a \leq |X|$  then  $\binom{X}{a}$  is the set of all  $a$ -sized subsets of  $X$ .
3. We identify  $\binom{X}{a}$  with the complete  $a$ -ary hypergraph on the set  $X$ . Hence we will use terms like *vertex* and *edge* when referring to  $\binom{X}{a}$ .
4. We will often have  $X \subseteq \mathbb{N}$  and a coloring  $COL : \binom{X}{a} \rightarrow Y$  ( $Y$  is either  $[c]$  or  $\omega$ ). We use the usual convention of using  $COL(x_1, \dots, x_a)$  for  $COL(\{x_1, \dots, x_a\})$ . We do not take this to mean that  $x_1 < \dots < x_a$ .

We define terms and then state the canonical Ramsey theorem (for graphs). It was first proven by Erdős and Rado [13]. The best known upper bounds on the canonical Ramsey numbers are due to Lefmann and Rödl [23].

**Def 2.2** Let  $COL : \binom{[n]}{2} \rightarrow \omega$ . Let  $V \subseteq [n]$ .

1. The set  $V$  is *homogenous* (henceforth *homog*) if for all  $x_1 < x_2$  and  $y_1 < y_2$

$$COL(x_1, x_2) = COL(y_1, y_2) \text{ iff } TRUE.$$

(Every edge in  $\binom{V}{2}$  is colored the same.)

2. The set  $V$  is *min-homogenous* (henceforth *min-homog*) if for all  $x_1 < x_2$  and  $y_1 < y_2$

$$COL(x_1, x_2) = COL(y_1, y_2) \text{ iff } x_1 = y_1.$$

3. The set  $V$  is *max-homogenous* (henceforth *max-homog*) if for all  $x_1 < x_2$  and  $y_1 < y_2$

$$COL(x_1, x_2) = COL(y_1, y_2) \text{ iff } x_2 = y_2.$$

4. The set  $V$  is *rainbow* if for all  $x_1 < x_2$  and  $y_1 < y_2$

$$COL(x_1, x_2) = COL(y_1, y_2) \text{ iff } (x_1 = y_1 \text{ and } x_2 = y_2).$$

(Every edge in  $\binom{V}{2}$  is colored differently.)

**Theorem 2.3** *For all  $k$  there exists  $n$  such that, for all colorings of  $\binom{[n]}{2}$  there is either a homog set of size  $k$ , a min-homog set of size  $k$ , a max-homog set of size  $k$ , or a rainbow set of size  $k$ . We denote the least value of  $n$  that works by  $ER(k)$ .*

We now state the asymmetric canonical Ramsey Theorem.

**Theorem 2.4** *For all  $k_1, k_2$  there exists  $n$  such that, for all colorings of  $\binom{[n]}{2}$ , there is either a homog set of size  $k_1$ , a min-homog set of size  $k_1$ , a max-homog set of size  $k_1$ , or a rainbow set of size  $k_2$ . We denote the least value of  $n$  that works by  $ER(k_1, k_2)$ .*

We will actually need a variant of the asymmetric canonical Ramsey Theorem which is weaker but gives better upper bounds.

**Def 2.5** Let  $COL : \binom{[n]}{2} \rightarrow \omega$ . Let  $V \subseteq [n]$ . The set  $V$  is *weakly homogenous* (henceforth *whomog*) if there is a way to linear order  $V$  (not necessarily the numerical order),

$$V = \{x_1, x_2, \dots, x_L\},$$

such that, for all for all  $1 \leq i \leq L - 3$ , for all  $i < j < k \leq L$ ,

$$COL(x_i, x_j) = COL(x_i, x_k).$$

Informally, the color of  $(x_i, x_j)$ , where  $i < j$ , depends only on  $i$ . (We intentionally have  $1 \leq i \leq L - 3$ . We do not care if  $COL(x_{L-2}, x_{L-1}) = COL(x_{L-2}, x_L)$ .)

**Note 2.6** When presenting a whomog set we will also present the needed linear order.

The following theorem follows from 2.4.

**Theorem 2.7** *For all  $k_1, k_2$  there exists  $n$  such that, for all colorings of  $\binom{[n]}{2}$  there is either a whomog set of size  $k_1$ , or a rainbow set of size  $k_2$ . We denote the least value of  $n$  that works by  $WER(k_1, k_2)$ .*

In Theorem 4.1 we will show

$$WER(k_1, k_2) \leq \frac{(Ck_2)^{6k_1-18}}{(\log k_2)^{2k_1-6}}.$$

### 3 Lemma to Help Obtain Rainbow Sets

The next definition and lemmas gives a way to get a rainbow set under some conditions.

**Def 3.1** Let  $COL : \binom{[m]}{2} \rightarrow \omega$ . If  $c$  is a color and  $v \in [m]$  then  $\deg_c(v)$  is the number of  $c$ -colored edges with an endpoint in  $v$ .

The following result is due to Alon, Lefmann, and Rödl [1].

**Lemma 3.2** Let  $m \geq 3$ .

1. Let  $COL : \binom{[m]}{2} \rightarrow \omega$  be such that, for all  $v \in [m]$  and all colors  $c$ ,  $\deg_c(v) \leq 1$ . Then there exists a rainbow set of size  $\geq \Omega((m \log m)^{1/3})$ .
2. There exists a coloring of  $\binom{[m]}{2}$  such that for all  $v \in [m]$  and all colors  $c$ ,  $\deg_c(v) \leq 1$  and all rainbow sets are of size  $\leq O((m \log m)^{1/3})$ .

The following easily follows:

**Lemma 3.3** Let  $m \geq 3$ . Let  $COL : \binom{[m]}{2} \rightarrow \omega$  be such that, for all  $v \in [m]$  and all colors  $c$ ,  $\deg_c(v) \leq 1$ . If  $m = \Omega(\frac{k^3}{\log k})$  then there exists a rainbow set of size  $k$ .

The following definitions and lemmas will be used to achieve the premise of Lemma 3.3

**Def 3.4** Let  $COL : \binom{[m]}{2} \rightarrow \omega$ . Let  $c$  be a color and let  $x \in [m]$ .

1.  $\deg_c(x)$  is the number of  $c$ -colored edges  $(x, y)$ .
2. A *bad triple* is a triple  $a, b, c$  such that  $a, b, c$  does not form a rainbow  $K_3$ .

The next two lemmas show us how to, in some cases, reduce the number of bad triples.

**Lemma 3.5** Let  $COL : \binom{[m]}{2} \rightarrow \omega$  be such that, for every color  $c$  and vertex  $v$ ,  $\deg_c(v) \leq d$ . Then the number of bad triples is less than  $\frac{dm^2}{6}$ .

**Proof:** We assume that  $d$  divides  $m - 1$ . We leave the minor adjustment needed in case  $d$  does not divide  $m - 1$  to the reader.

Let  $b$  be the number of bad triples. We upper bound  $b$  by summing over all  $v$  that are the point of the triple with two same-colored edges coming out of it. This actually counts each triple thrice. Hence we have

$$\begin{aligned} 3b &\leq \sum_{v \in [m]} \sum_{c \in \mathbf{N}} \text{Num of bad triples } \{v, u_1, u_2\} \text{ with } COL(v, u_1) = COL(v, u_2) = c \\ &\leq \sum_{v \in [m]} \sum_{c \in \mathbf{N}} \binom{\deg_c(v)}{2} \end{aligned}$$

We bound the inner summation. Since  $v$  is of degree  $m - 1$  we can renumber the colors as  $1, 2, \dots, m - 1$ . Since  $\sum_{c=1}^{m-1} \deg_c(v) = m - 1$  and  $(\forall c)[\deg_c(v) \leq d]$  the sum  $\sum_{c=1}^{m-1} \binom{\deg_c(v)}{2}$  is

maximized when  $d = \deg_1(v) = \deg_2(v) = \dots = \deg_{(m-1)/d}(v)$  and the rest of the  $\deg_c(v)$ 's are 0. Hence

$$3b \leq \sum_{v \in [m]} \sum_{c=1}^{m-1} \binom{\deg_c(v)}{2} \leq \sum_{v \in [m]} \sum_{c=1}^{(m-1)/d} \binom{d}{2} < \frac{dm^2}{2}.$$

Hence  $b \leq \frac{dm^2}{6}$ . ■

**Lemma 3.6** *Let  $COL : \binom{[m]}{2} \rightarrow \omega$  be such that there are  $\leq b$  bad triples. Let  $1 \leq m' \leq m$ . There exists an  $m'$ -sized set of vertices with  $\leq b\left(\frac{m'}{m}\right)^3$  bad triples.*

**Proof:** Pick a set  $X$  of size  $m'$  at random. Let  $E$  be the expected number of bad triples. Note that

$$E = \sum_{\{v_1, v_2, v_3\} \text{ bad}} \text{Prob that } \{v_1, v_2, v_3\} \subseteq X.$$

Let  $\{v_1, v_2, v_3\}$  be a bad triple. The probability that all three nodes are in  $X$  is bounded by

$$\frac{\binom{m-3}{m'-3}}{\binom{m}{m'}} \leq \frac{m'(m'-1)(m'-2)}{m(m-1)(m-2)} \leq \left(\frac{m'}{m}\right)^3.$$

Hence the expected number of bad triples is  $\leq b\left(\frac{m'}{m}\right)^3$ . Therefore there must exist some  $X$  that has  $\leq b\left(\frac{m'}{m}\right)^3$  bad triples. ■

**Note 3.7** The above theorem presents the user with an interesting tradeoff. She wants a large set with few bad triples. If  $m'$  is large then you get a large set, but it will have many bad triples. If  $m'$  is small then you won't have many bad triples, but  $m'$  is small. We will need a Goldilocks- $m'$  that is just right.

## 4 The Asymmetric Weak Canonical Ramsey Theorem

**Theorem 4.1** *There exists  $C$  such that, for all  $k_1, k_2$ ,*

$$WER(k_1, k_2) \leq \frac{(Ck_2)^{6k_1-18}}{(\log k_2)^{2k_1-6}}.$$

**Proof:**

Let  $n, m, m', m'', \delta$  be parameters to be determined later. They will be functions of  $k_1, k_2$ . Let  $COL : \binom{[n]}{2} \rightarrow \omega$ .

**Intuition:** In the usual proofs of Ramsey's Theorem (for two colors) we take a vertex  $v$  and see which of  $\deg_{RED}(v)$  or  $\deg_{BLUE}(v)$  is large. One of them must be at least half of the size of the vertices still in play. Here we change this up:

- Instead of taking a particular vertex  $v$  we ask if there is *any*  $v$  and *any* color  $c$  such that  $\deg_c(v)$  is large.
- What is large? Similar to the proof of Ramsey's theorem it will be a fraction of what is left. Unlike the proof of Ramsey's theorem this fraction,  $\delta$ , will depend on  $k_2$ .
- In the proof of Ramsey's theorem we were guaranteed that one of  $\deg_{RED}(v)$  or  $\deg_{BLUE}(v)$  is large. Here we have no such guarantee. We may fail. In that case something else happens and leads to a rainbow set!

**CONSTRUCTION****Phase 1:****Stage 0:**

1.  $V_0 = \emptyset$ .
2.  $N_0 = [n]$ .
3.  $COL'$  is not defined on any points.

**Stage  $i$ :** Assume that  $V_{i-1} = \{x_1, \dots, x_{i-1}\}$ ,  $c_1, \dots, c_{i-1}$ , and  $N_{i-1}$  are already defined.

If there exists  $x \in N_{i-1}$  and  $c$  a color such that  $\deg_c(x) \geq \delta N_{i-1}$  then do the following:

$$\begin{aligned} V_i &= V_{i-1} \cup \{x\} \\ N_i &= \{v \in N_{i-1} : COL(x, v) = c\} \\ x_i &= x \\ c_i &= c \end{aligned}$$

Note that  $|N_i| \geq \delta |N_{i-1}|$ , so  $|N_i| \geq \delta^i n$ , and  $|V_i| = i$ . If  $i = k_1 - 3$  then goto Phase 2.

If no such  $x, c$  exist then goto Phase 3. In this case we formally regard the jump to Phase 3 as happening in stage  $i - 1$  since nothing has changed.

**End of Phase 1**

**Phase 2:** Since we are in Phase 2  $i = k_1 - 3$ . Let

$$V = V_{k_1-3} = \{x_1, x_2, \dots, x_{k_1-3}\}.$$

(This is the order the elements came into  $V$ , not the numeric order.) By construction  $V$  is a whomog set of size  $k_1 - 3$ . Note that, for all elements  $x \in N_{k_1-3}$ ,  $COL(x_i, x) = c_i$ .

We need  $|N_{k_1-3}| \geq 3$  (you will see why soon). Since  $|N_{k_1-3}| \geq \delta^{k_1-3}n$  we satisfy  $|N_{k_1-3}| \geq 3$  by imposing the constraint

$$n \geq \frac{3}{\delta^{k_1-3}}.$$

Let  $x_{k_1-2}$ ,  $x_{k_1-1}$ , and  $x_{k_1}$  be three points from  $N_{k_1-3}$ . Let  $H$  be (in this order)

$$H = \{x_1, x_2, \dots, x_{k_1-3}, x_{k_1-2}, x_{k_1-1}, x_{k_1}\}.$$

$H$  is clearly whomog. (Recall that in a whomog set of size  $k_1$  we do not care if  $COL(x_{k_1-2}, x_{k_1-1}) = COL(x_{k_1-2}, x_{k_1})$ .)

**End of Phase 2**

**Phase 3:** Since we are in Phase 2  $i \leq k_1 - 4$ . Let  $N = N_i$ .

$$|N| \geq \delta^i n \geq \delta^{k_1-4} n.$$

We will need  $|N| \geq m$  since we will find a rainbow subset of  $N$  and need  $N$  to be big in the first place so that the rainbow subset is of size at least  $k_2$ . Hence we impose the constraint

$$n \geq \frac{m}{\delta^{k_1-4}}.$$

Recall that we also imposed the constraint  $n \geq \frac{3}{\delta^{k_1-3}}$ . To satisfy both of these constraints we impose the following two constraints:

$$m = \frac{3}{\delta}$$

and

$$n = \frac{3}{\delta^{k_1-3}}.$$

Let  $|N| = m_0$ . We have no control over  $m_0$ . All we will know is that  $m \leq m_0 \leq n$ . Later on  $m_0$  will cancel out of calculations and hence we can set other parameters independent of it.

Let  $COL$  be the coloring restricted to  $\binom{N}{2}$ . We can assume the colors are a subset of  $\{1, \dots, \binom{m_0}{2}\}$ . Since we are in Phase 3 we know that, for all  $v \in N$ , for all colors  $c$ ,  $\deg_c(v) \leq \delta m_0$ . Hence, by Lemma 3.5, there are at most

$$\frac{\delta m_0 \times m_0^2}{6} \leq \delta m_0^3$$

bad triples (we ignore the denominator of 6 since it makes later calculations easier and only affects the constant).

By Lemma 3.6 there exists  $X \subseteq N$  of size  $m'$  that has

$$b < \delta m_0^3 \times \left(\frac{m'}{m_0}\right)^3 = \delta (m')^3$$



bad triples. Note that the bound on  $b$  is independent of  $m_0$ .

We set  $m'$  such that the number of bad triples is so small that we can just remove one point from each to obtain a set  $X$  of size  $m'$  with *no* bad triples.

Since the number of bad triples is  $\leq \delta(m')^3$  we need

$$m' - \delta(m')^3 \geq m''.$$

Hence we impose the constraint

$$\delta = \frac{m' - m''}{(m')^3}.$$

We will now set the parameters. Since we will use Lemma 3.3 it would be difficult to optimize the parameters. Hence we pick parameters that are easy to work with.

We will use Lemma 3.3 on  $X$  to obtain a rainbow set of size  $k_2$ . Hence we take

$$m'' = \frac{Ak_2^3}{\log k_2}$$

where  $A$  is chosen to (1) make  $m''$  large enough to satisfy the premise of Lemma 3.3, (2) make  $m''$  an integer, and (3) make  $m', m, n$ , which will be functions of  $m''$ , integers.

We take

$$m' = 1.5m''. \text{ This is the value that minimize } \delta \text{ though this does not matter.}$$

With this value of  $m'$  we obtain

$$\delta = \frac{m' - m''}{(m'')^3} = \frac{1}{B(m'')^2}$$

where  $B$  is an appropriate constant. Our constraints force

$$\begin{aligned} m &= \frac{3}{\delta} \\ n &= \frac{3}{\delta^{k_1-3}} = 3(Bm'')^{2(k_1-3)} = \frac{(Ck_2)^{6k_1-18}}{(\log k_2)^{2k_1-6}} \end{aligned}$$

where  $C$  is an appropriate constant. ■

## 5 Lemmas from Geometry

**Def 5.1** Let  $d \in \mathbb{N}$ .

1. If  $p, q \in \mathbb{R}^d$  then let  $|p - q|$  be the Euclidean distance between  $p$  and  $q$ .
2. Let  $p_1, \dots, p_n$  be points in  $\mathbb{R}^d$ .  $(p_1, \dots, p_n)$  is a *cool sequence* if, for all  $1 \leq i \leq n - 3$ , for all  $i < j \leq n$ ,  $|p_i - p_j|$  is determined solely by  $p_i$ . (Formally: for all  $1 \leq i \leq n - 3$  there exists  $L_i$  such that, for all  $i + 1 \leq j \leq n$ ,  $|p_i - p_j| = L_i$ .) We intentionally have  $1 \leq i \leq n - 3$ . We do not care if  $|p_{n-2} - p_{n-1}| = |p_{n-2} - p_n|$ .

3. The sphere with center  $x \in \mathbb{R}^{d+1}$  and radius  $r \in \mathbb{R}^+$  is the set

$$\{y \in \mathbb{R}^{d+1} : |x - y| = r\}.$$

If the sphere is completely contained in an  $(n + 1)$ -dimensional plane then the sphere is called an  $n$ -sphere.

Note that if  $(p_1, \dots, p_n)$  is cool then  $(p_2, \dots, p_n)$  is cool. We use this implicitly without mention.

The following lemma is well known.

**Lemma 5.2** *Let  $S$  be a  $d$ -sphere. Let  $x \in S$  and  $r \in \mathbb{R}^+$ . The set*

$$\{y \in S : |x - y| = r\}$$

*is either an  $(d - 1)$ -sphere or is empty.*

**Lemma 5.3** *For all  $d \geq 0$  there does not exist a cool sequence  $p_1, \dots, p_{d+3}$  on a  $d$ -sphere.*

**Proof:** We prove this by induction on  $d$ .

**Base Case  $d = 0$ :** Assume, by way of contradiction, that  $(p_1, p_2, p_3)$  form a cool sequence on a 0-sphere. A 0-sphere is a set of two points, hence this is impossible. (Note that being a cool sequence did not constraint  $(p_1, p_2, p_3)$  at all.)

**Induction Hypothesis:** The theorem holds for  $d - 1$ .

**Induction Step:** We prove the theorem for  $d$ . We may assume  $d \geq 1$ . Assume, by way of contradiction, that  $(p_1, \dots, p_{d+2})$  form a cool sequence on an  $d$ -sphere. Since  $|p_1 - p_2| = |p_1 - p_3| = \dots = |p_1 - p_{d+2}|$  we know, by Lemma 5.2, that  $p_2, p_3, \dots, p_{d+2}$  are on an  $(d - 1)$ -sphere. Since  $p_2, \dots, p_{d+2}$  is a cool sequence this is impossible by the induction hypothesis.

■

**Note 5.4** The following related statement is well known: *if there are  $d + 2$  points in  $\mathbb{R}^d$  then it is not the case that all  $\binom{d+2}{2}$  distances are the same.* We have not been able to locate this result in an old fashion journal (perhaps its behind a paywall); however, there is a proof at mathoverflow.net here:

<http://mathoverflow.net/questions/30270/maximum-number-of-mutually-equidistant-points-in-an-n-dimensional-euclidean-space>

**Lemma 5.5** *Let  $d \in \mathbb{N}$ . Let  $p_1, \dots, p_n$  be points in  $\mathbb{R}^d$ . Color  $\binom{[n]}{2}$  via  $COL(i, j) = |p_i - p_j|$ . This coloring has no whomog set of size  $d + 3$ .*

**Proof:** Assume, by way of contradiction, that there exists a whomog set of size  $d + 3$ . By renumbering we can assume the whomog set is  $[d + 3]$ . Clearly  $p_1, \dots, p_{d+3}$  form a cool sequence. Note that our not-caring about  $COL(d + 1, d + 2) = COL(d + 1, d + 3)$  in the definition of whomog is reflected in our not-caring about  $|p_{d+1} - p_{d+2}| = |p_{d+1} - p_{d+3}|$  in the definition of a cool sequence.

Since  $|p_1 - p_2| = |p_1 - p_3| = \dots = |p_1 - p_{d+3}|$ ,  $p_2, \dots, p_{d+3}$  are on the  $(d - 1)$ -sphere (centered at  $p_1$ ). This contradicts Lemma 5.3. ■

## 6 Lower Bound on $h_{2,d}(n)$

We defined  $WER(k_1, k_2)$  in terms of colorings with co-domain  $\omega$ . In our application we will actually use colorings with co-domain  $\mathbb{R}^+$ . The change in our results to accommodate this is only a change of notation. Hence we use our lower bounds on  $WER(k_1, k_2)$  in this context without mention.

**Theorem 6.1** *For all  $d \geq 1$ ,  $h_{2,d}(n) = \Omega((n^{1/(6d)}(\log n)^{1/3})/d^{1/3})$ .*

**Proof:** Let  $P = \{p_1, \dots, p_n\}$  be  $n$  points in  $\mathbb{R}^d$ . Let  $COL : \binom{[n]}{2} \rightarrow \mathbb{R}$  defined by  $COL(i, j) = |p_i - p_j|$ .

Let  $k$  be the largest integer such that  $n \geq WER(d+3, k)$ . By Theorem 4.1 it will suffice to take  $k = \Omega((n^{1/(6d)}(\log n)^{1/3})/d^{1/3})$ . By the definition of  $WER_3(d+3, k)$  there is either a whomog set of size  $d+3$  or a rainbow set of size  $k$ . By Lemma 5.5 there cannot be such a whomog set, hence must be a rainbow set of size  $k$ . ■

## 7 Lower Bounds on $h_{3,2}$ and $h_{3,3}$

For the problem of  $h_{2,d}$  we used (1) upper bounds on the asymmetric weak canonical Ramsey theorem and (2) a geometric lemma. Here we will use the same approach though our version of the asymmetric weak canonical Ramsey theorem does not involve reordering the vertices.

### 7.1 The Asymmetric 3-ary Canonical Ramsey Theorem

**Def 7.1** Let  $COL : \binom{[n]}{a} \rightarrow \omega$ . Let  $V \subseteq [n]$ .

1. Let  $I \subseteq [a]$ . The set  $V$  is *I-homogenous* (henceforth *I-homog*) if for all  $x_1 < \dots < x_a \in \binom{[n]}{a}$  and  $y_1 < \dots < y_a \in \binom{[n]}{a}$ ,

$$(\forall i \in I)[x_i = y_i] \text{ iff } COL(x_1, \dots, x_a) = COL(y_1, \dots, y_a).$$

Informally, the color of an element of  $\binom{[n]}{a}$  depends exactly on the coordinates in  $I$ .

2. The set  $V$  is *rainbow* if every edge in  $\binom{V}{a}$  is colored differently. Note that this is just an *I-homog* set where  $I = [a]$ .

We will need the asymmetric hypergraph Ramsey numbers and  $a$ -ary Erdős-Rado numbers.

**Def 7.2** Let  $a \geq 1$ . Let  $k_1, k_2, \dots, k_c \geq 1$ .

1. Let  $COL : \binom{[n]}{a} \rightarrow [c]$ . (Note that there is a bound on the number of colors.) Let  $V \subseteq [n]$ . The set  $V$  is *homog with color  $i$*  if  $COL$  restricted to  $\binom{V}{a}$  always returns  $i$ .

2.  $R_a(k_1, k_2, \dots, k_c)$  is the least  $n$  such that, for all  $COL : \binom{[n]}{a} \rightarrow [c]$ , there exists  $1 \leq i \leq c$  and a homog set of size  $k_i$  with color  $i$ .  $R_a(k_1, k_2, \dots, k_c)$  is known to exist by the hypergraph Ramsey theorem.
3.  $ER_a(k_1, k_2)$  is the least  $n$  such that, for all  $COL : \binom{[n]}{a} \rightarrow \omega$ , there exists either (1) an  $I \subset [a]$  (note that this is a proper subset) and an  $I$ -homog set of size  $k_1$ , or (2) a rainbow set of size  $k_2$ .  $ER_a(k_1, k_2)$  is known to exist by the  $a$ -ary canonical Ramsey theorem.

**Def 7.3** Let  $a \geq 3$ .

1. Let  $COL : \binom{[n]}{a} \rightarrow \omega$ . Let  $V \subseteq [n]$ . Let  $I \subset [a]$ . The set  $V$  is  *$I$ -weakly homogenous* (henceforth  *$I$ -whomog*) if for all  $x_1, \dots, x_a, y_1, \dots, y_a \in [n]$

$$(\forall i \in I)[x_i = y_i] \implies COL(x_1, \dots, x_a) = COL(y_1, \dots, y_a).$$

(Note that this differs slightly from the  $a = 2$  case in that we do not change around the ordering.)

2. Let  $COL : \binom{[n]}{a} \rightarrow \omega$ . Let  $V \subseteq [n]$ . The set  $V$  is *weakly homogenous* (henceforth *whomog*) if there is an  $I \subset [a]$  (note that this is proper subset) such that  $V$  is  $I$ -whomog.
3. Let  $k_1, k_2 \in \mathbb{N}$ . We denote the least  $n$  such that, for all  $COL : \binom{[n]}{a} \rightarrow \omega$ , there is either a whomog set of size  $k_1$  or a rainbow set of size  $k_2$ , by  $WER_a(k_1, k_2)$ .  $WER_a(k_1, k_2)$  is known to exist by the  $a$ -ary canonical Ramsey theorem.

**Note 7.4** Note that if a set is  $\{1\}$ -whomog then its also  $\{1, 2\}$ -whomog.

A modification of the bound on  $ER_3(k)$  by Lefmann and Rödl [22] yields

$$ER_3(k_1, k_2) \leq R_4(6, 6, 6, 6, k_1, k_1, k_1, k_1, \left\lceil \frac{k_1^3}{4} \right\rceil, \left\lceil \frac{k_1^3}{4} \right\rceil, 2k_1^3, \left\lceil \frac{k_2^5}{36} \right\rceil).$$

We get better bounds on  $WER_3(k_1, k_2)$ .

We first need the  $k = 3$  case of Lemma 3 of [22] which we state:

**Lemma 7.5** Let  $COL : \binom{X}{3} \rightarrow \omega$  be such that, for all  $S, T \in \binom{X}{3}$ , with  $|S \cap T| = 2$ ,  $COL(S) \neq COL(T)$ . Then there exists a rainbow set of size  $\geq \Omega(|X|^{1/5})$ .

**Lemma 7.6**  $WER_3(k_1, k_2) \leq R_4(k_1, k_1 + 2, k_1 + 2, k_1, k_1 + 2, k_1, k_2^5)$

**Proof:** Let  $n = R_4(k_1, k_1 + 2, k_1 + 2, k_1, k_1 + 2, k_1, k_2^5)$ .

We are given  $COL : \binom{[n]}{3} \rightarrow \omega$ . We use  $COL$  to obtain a  $COL' : \binom{[n]}{4} \rightarrow [7]$ . We will use the (ordinary) 3-ary Ramsey theorem.

We define  $COL'(x_1 < x_2 < x_3 < x_4)$  by looking at  $COL$  on all  $\binom{4}{3}$  triples of  $\{x_1, x_2, x_3, x_4\}$  and see how their colors compare to each other.

For each case we assume the negation of all the prior cases. In each case, we indicate what happens if this is the color of the infinite homog set.

In all the cases below we use the following notation: if we are referring to a set  $X$  and  $x \in X$  then  $x^+$  is the next element of  $X$  after  $x$ .

1. If  $COL(x_1, x_2, x_3) = COL(x_1, x_2, x_4)$  then  $COL'(x_1, x_2, x_3, x_4) = 1$ . Assume  $X$  is a homog set of size  $k_1$  with color 1. Clearly  $X$  is  $\{1, 2\}$ -whomog for  $COL$ .
2. If  $COL(x_1, x_2, x_3) = COL(x_1, x_3, x_4)$  then  $COL'(x_1, x_2, x_3, x_4) = 2$ . Assume  $X$  is a homog set of size  $k_1 + 2$  with color 2. Let  $z_1, z_2$  be the largest two elements of  $X$ . We show that  $X - \{z_1, z_2\}$  is  $\{1\}$ -whomog for  $COL$ . Assume (1)  $x_1 < x_2 < x_3$ , (2)  $x_1 < y_2 < y_3$ , and (3)  $x_1, x_2, x_3, y_2, y_3 \in X - \{z_1, z_2\}$ . We need  $COL(x_1, x_2, x_3) = COL(x_1, y_2, y_3)$ .

$$COL(x_1, x_2, x_3) = COL(x_1, x_3, x_3^+) = \cdots = COL(x_1, z_1, z_2)$$

and

$$COL(x_1, y_2, y_3) = COL(x_1, y_3, y_3^+) = \cdots = COL(x_1, z_1, z_2).$$

Hence they equal each other.

3. If  $COL(x_1, x_2, x_3) = COL(x_2, x_3, x_4)$  then  $COL'(x_1, x_2, x_3, x_4) = 3$ . Assume  $X$  is a homog set of size  $k_1 + 2$  with color 3. Let  $z_1, z_2$  be the largest two elements of  $X$ . We show that  $X - \{z_1, z_2\}$  is  $\emptyset$ -whomog for  $COL$  (all edges are the same color). Note that all triples of the form  $(x, x^+, x^{++})$  have the same color. Denote that color *RED*. Assume (1)  $x_1 < x_2 < x_3$ , (2)  $y_1 < y_2 < y_3$ , and (3)  $x_1, x_2, x_3, y_1, y_2, y_3 \in X - \{z_1, z_2\}$ . We need  $COL(x_1, x_2, x_3) = COL(y_1, y_2, y_3)$ . Note that

$$COL(x_1, x_2, x_3) = COL(x_2, x_3, x_3^+) = COL(x_3, x_3^+, x_3^{++}) = RED.$$

By the same reasoning  $COL(y_1, y_2, y_3) = RED$ . (Note that it is possible that  $x_3^+, x_3^{++}, y_3^+, y_3^{++} \in \{z_1\}$  or  $x_3^{++}, y_3^{++} \in \{z_2\}$ .)

4. If  $COL(x_1, x_2, x_4) = COL(x_1, x_3, x_4)$  then  $COL'(x_1, x_2, x_3, x_4) = 4$ . Assume  $X$  is a homog set of size  $k_1$  with color 4. Clearly  $X$  is  $\{1, 3\}$ -whomog for  $COL$ .
5. If  $COL(x_1, x_2, x_4) = COL(x_2, x_3, x_4)$  then  $COL'(x_1, x_2, x_3, x_4) = 5$ . Assume  $X$  is a homog set of size  $k_1 + 2$  with color 5. Let  $z_1, z_2$  be the smallest elements of  $X$ . Then  $X - \{z_1, z_2\}$  is  $\{3\}$ -whomog for  $COL$  by the same reasoning as in part 2.
6. If  $COL(x_1, x_3, x_4) = COL(x_2, x_3, x_4)$  then  $COL'(x_1, x_2, x_3, x_4) = 6$ . Assume  $X$  is a homog set of size  $k_1$  with color 6. Clearly  $X$  is  $\{2, 3\}$ -whomog for  $COL$ .

7. If none of the above happen then  $COL'(x_1, x_2, x_3, x_4) = 7$ . Assume  $X$  is a homog set of size  $k_2^5$  with color 7.  $COL$  restricted to  $\binom{X}{3}$  satisfies the premise of Lemma 7.5: If  $S, T \in \binom{X}{3}$  with  $|S \cap T| = 2$  then since  $COL'(S \cup T) \notin \{1, 2, 3, 4, 5, 6\}$ ,  $COL(S) \neq COL(T)$ . By Lemma 7.5 there exists a rainbow set of size  $|X|^{1/5} \geq k_2$ .

■

**Lemma 7.7** *Let  $a \geq 3$ ,  $c \geq 2$ , and  $k_1, \dots, k_c \geq 1$ . Let  $P = k_1 \cdots k_{c-1}$  and  $S = k_1 + \cdots + k_{c-1}$ .*

1.  $R_a(k_1, k_2, \dots, k_c) \leq c^{R_{a-1}(k_1-1, k_2-1, \dots, k_c-1)^{a-1}}.$

2. Let

$$Z = \{\sigma \in [c]^* : \text{for all } i \in [c], \sigma \text{ contains at most } k_i - 1 \text{ } i\text{'s}\}.$$

Then

$$\sum_{\sigma \in Z} |\sigma| \leq P(k_c + S)^{S+2}.$$

3.  $R_3(k_1, \dots, k_c) \leq c^{P(k_c+S)^{S+2}}.$

4.  $R_4(k_1, \dots, k_c) \leq c^{c^{3P(k_c+S-c)^{S+2-c}}}.$

5. For almost all  $k$ ,  $WER_3(e, k) \leq 2^{2^{k^{30e+6}}}.$

6.  $WER_3(6, k) \leq 2^{2^{k^{186}}}.$

7.  $WER_3(13, k) \leq 2^{2^{k^{396}}}.$

**Proof:**

1) Erdős-Rado [14, 19, 20] showed that  $R_a(k, k) \leq 2^{\binom{R_{a-1}(k-1, k-1)+1}{a-1}} + a - 2$ . This can be modified to show

$$R_a(k_1, k_2, \dots, k_c) \leq c^{\binom{R_{a-1}(k_1-1, \dots, k_c-1)}{a-1} + a - 2}.$$

Our result easily follows.

2) Clearly

$$\begin{aligned} \sum_{\sigma \in Z} |\sigma| &= \sum_{j_1=0}^{k_1-1} \cdots \sum_{j_c=0}^{k_c-1} (j_1 + \cdots + j_c) \frac{(j_1 + \cdots + j_c)!}{j_1! \cdots j_c!} \leq \sum_{j_1=0}^{k_1-1} \cdots \sum_{j_c=0}^{k_c-1} (k_c + S) \frac{(k_c + S)!}{k_c!} \\ &\leq P \sum_{j_c=0}^{k_c-1} (k_c + S)^{S+1} \leq P k_c (k_c + S)^{S+1} \leq P (k_c + S)^{S+2} \end{aligned}$$

3) Conlon, Fox, and Sudakov [4] have the best known upper bounds on  $R_3(k, k)$ . Gasarch, Parrish, Sadow [19] have done a straightforward analysis of their proof to extend it to  $c$  colors. A modification of that proof yields

$$R_3(k_1, \dots, k_c) \leq c^{\sum_{\sigma \in Z} |\sigma|}.$$

Our result follows.

4) This follows from parts 1 and 3. We could obtain a better result by replacing  $P$  by  $(k_1 - 1) \cdots (k_c - 1)$  but that would not help us later.

5) By Lemma 7.6

$$WER_3(e, k) \leq R_4(e, e + 2, e + 2, e, e + 2, e, k^5).$$

Let  $s(e) = e + (e + 2) + (e + 2)e + (e + 2) + e = 6e + 6$ . Let  $p(e)$  be the product of these terms. By parts 2 and 5, for  $k$  large, we have the following.

$$WER_3(e, k) \leq 7^{7^{3p(e)(k^5 + s(e) - 7)^{s(e) - 5}}} \leq 7^{7^{3p(e)(2k^5)^{s(e) - 5}}} \leq 7^{7^{(6p(e)k^5)^{s(e) - 5}}} \leq 2^{2^{(36p(e)k^5)^{s(e) - 5}}}$$

Let  $f(e) = (36p(e))^{s(e) - 5}$ . Then we have

$$WER_3(e, k) \leq 2^{2^{f(e)k^{5s(e) - 25}}} \leq 2^{2^{k^{5s(e) - 24}}} \leq 2^{2^{k^{30e + 6}}}.$$

6) This follows from part 5.

7) This follows from part 5.

■

## 7.2 Geometric Lemmas

**Def 7.8** Let  $d \in \mathbb{N}$ . If  $p, q, r \in \mathbb{R}^d$  then let  $AREA(p, q, r)$  be the area of the triangle with vertices  $p, q, r$ .

The next lemma is Lemma 4 of [6] whose proof is in the appendix of that paper. They credit [16], which is unavailable, with the proof.

**Lemma 7.9** *Let  $C_1, C_2, C_3$  be three cylinders with no pair of parallel axis. Then  $C_1 \cap C_2 \cap C_3$  consists of at most 8 points.*

### Lemma 7.10

1. Let  $p_1, \dots, p_n$  be points in  $\mathbb{R}^2$ , no three collinear. Color  $\binom{[n]}{3}$  via  $COL(i, j, k) = AREA(p_i, p_j, p_k)$ . This coloring has no whomog set of size 6.
2. Let  $p_1, \dots, p_n$  be points in  $\mathbb{R}^3$ , no three collinear. Color  $\binom{[n]}{3}$  via  $COL(i, j, k) = AREA(p_i, p_j, p_k)$ . This coloring has no whomog set of size 13.

**Proof:**

1) Assume, by way of contradiction, there exists an  $I$ -whomog set of size 6. By renumbering we can assume the  $I$ -whomog set is  $\{1, 2, 3, 4, 5, 6\}$ .

**Case 1:**  $I = \{1\}$ ,  $\{1, 2\}$ , or  $\{2\}$ .

We have  $AREA(p_1, p_2, p_4) = AREA(p_1, p_2, p_5)$ . Thus  $p_4$  and  $p_5$  are either on a line parallel to  $p_1p_2$  or are on different sides of  $p_1p_2$ . In the later case the midpoint of  $p_4p_5$  is on  $p_1p_2$ .

We have  $AREA(p_1, p_3, p_4) = AREA(p_1, p_3, p_5)$ . Thus  $p_4$  and  $p_5$  are either on a line parallel to  $p_1p_3$  or are on different sides of  $p_1p_3$ . In the later case the midpoint of  $p_4p_5$  is on  $p_1p_3$ .

We have  $AREA(p_2, p_3, p_4) = AREA(p_2, p_3, p_5)$ . Thus  $p_4$  and  $p_5$  are either on a line parallel to  $p_2p_3$  or are on different sides of  $p_2p_3$ . In the later case the midpoint of  $p_4p_5$  is on  $p_2p_3$ .

One of the following must happen.

- Two of these cases have  $p_4, p_5$  on the same side of the line. We can assume that  $p_4, p_5$  are on a line parallel to both  $p_1p_2$  and  $p_1p_3$ . Since  $p_1, p_2, p_3$  are not collinear there is no such line.
- Two of these cases have  $p_4, p_5$  on opposite sides of the line. We can assume that the midpoint of  $p_4p_5$  is on both  $p_1p_2$  and  $p_1p_3$ . Since  $p_1, p_2, p_3$  are not collinear the only point on both  $p_1p_2$  and  $p_1p_3$  is  $p_1$ . So the midpoint of  $p_4, p_5$  is  $p_1$ . Thus  $p_4, p_1, p_5$  are collinear which is a contradiction.

Note that for  $I = \{1\}$ ,  $\{1, 2\}$ , or  $\{2\}$  we used the line-point pairs

$$\{p_1p_2, p_1p_3, p_2p_3\} \times \{p_4, p_5\}.$$

For the rest of the cases we just specify which line-point pairs to use.

**Case 2:**  $I = \{3\}$  or  $\{2, 3\}$ . Use

$$\{p_4p_5, p_3p_5, p_3p_4\} \times \{p_1, p_2\}.$$

**Case 3:**  $I = \{1, 3\}$  Use

$$\{p_1p_4, p_1p_5, p_1p_6\} \times \{p_2, p_3\}.$$

This is the only case that needs 6 points.

2) Assume, by way of contradiction, that there exists an  $I$ -whomog set of size 13. By renumbering we can assume the  $I$ -whomog set is  $\{1, \dots, 13\}$ .

**Case 1:**  $I = \{1\}$ ,  $\{1, 2\}$ , or  $\{2\}$ .

We have  $AREA(p_1, p_2, p_4) = AREA(p_1, p_2, p_5) = \dots = AREA(p_1, p_2, p_{12})$ . Hence  $p_4, \dots, p_{12}$  are all on a cylinder with axis  $p_1p_2$ .



We have  $AREA(p_1, p_3, p_4) = AREA(p_1, p_3, p_5) = \dots = AREA(p_1, p_3, p_{12})$ . Hence  $p_4, \dots, p_{12}$  are all on a cylinder with axis  $p_1p_3$ .

We have  $AREA(p_2, p_3, p_4) = AREA(p_2, p_3, p_5) = \dots = AREA(p_2, p_3, p_{12})$ . Hence  $p_4, \dots, p_{12}$  are all on a cylinder with axis  $p_2p_3$ .

Since  $p_1, p_2, p_3$  are not collinear the three cylinders mentioned above satisfy the premise of Lemma 7.9. By that lemma there are at most 8 points in the intersection of the three cylinders. However, we just showed there are 9 such points. Contradiction.

Note that for  $I = \{1\}$ ,  $\{1, 2\}$ , or  $\{2\}$  we used the line-point pairs

$$\{p_1p_2, p_1p_3, p_2p_3\} \times \{p_4, \dots, p_{12}\}.$$

For the rest of the cases we just specify which line-point pairs to use.

**Case 2:**  $I = \{3\}$  or  $\{2, 3\}$ . Use

$$\{p_{11}p_{12}, p_{10}p_{12}, p_{10}p_{11}\} \times \{p_1, \dots, p_9\}.$$

**Case 3:**  $I = \{1, 3\}$  Use

$$\{p_1p_{11}, p_1p_{12}, p_1p_{13}\} \times \{p_2, \dots, p_{10}\}.$$

This is the only case that needs 13 points.

■

### 7.3 Lower Bounds on $h_{3,2}(n)$ and $h_{3,3}(n)$

#### Theorem 7.11

1.  $h_{3,2}(n) \geq \Omega((\log \log n)^{1/186})$ .
2.  $h_{3,3}(n) \geq \Omega((\log \log n)^{1/396})$ .

#### Proof:

a) Let  $P = \{p_1, \dots, p_n\}$  be  $n$  points in  $\mathbb{R}^2$ . Let  $COL$  be the coloring of  $\binom{[n]}{3}$  defined by  $COL(i, j, k) = AREA(p_i, p_j, p_k)$ .

Let  $k$  be the largest integer such that

$$n \geq WER_3(6, k).$$

By Lemma 7.7.6 it will suffice to take  $k = \Omega((\log \log n)^{1/186})$ . By the definition of  $WER_3(6, k)$  there is either a whomog set of size 6 or a rainbow set of size  $k$ . By Lemma 7.10.a there cannot be such a whomog set, hence must be a rainbow set of size  $k$ .

b) Let  $P = \{p_1, \dots, p_n\}$  be  $n$  points in  $\mathbb{R}^3$ . Let  $COL$  be the coloring of  $\binom{[n]}{3}$  defined by  $COL(i, j, k) = AREA(p_i, p_j, p_k)$ .

Let  $k$  be the largest integer such that

$$n \geq WER_3(13, k).$$

By Lemma 7.7.7 it will suffice to take  $k = \Omega((\log \log n)^{1/396})$ . By the definition of  $WER_3(13, k)$  there is either a whomog set of size 13 or a rainbow set of size  $k$ . By Lemma 7.10.b there cannot be such a whomog set, hence must be a rainbow set of size  $k$ . ■

Shelah [24] has shown that  $ER_3(k) \leq 2^{2^{p(k)}}$  for some polynomial  $p(k)$ . We suspect that this result can be modified to obtain  $ER_3(e, k) \leq 2^{k^{f(e)}}$  for some function  $f$ . If this is the case then there exists constants  $c_2, c_3$  such that  $h_{3,2}(n) \geq \Omega((\log n)^{c_2})$  and  $h_{3,3}(n) \geq \Omega((\log n)^{c_3})$ . Furthermore, we believe that better upper bounds can be obtained for  $WER_3(e, k)$  which will lead to larger values of  $c_2$  and  $c_3$ .

For  $d \geq 4$  we need the modification of Shelah's result and also some geometric lemmas. We believe both are true, though the geometric lemmas look difficult. Hence we believe that, for all  $d$ , there exists  $c_d$  such that  $h_{3,d}(n) \geq \Omega((\log n)^{c_d})$ .

## 8 $h_{a,d}(\alpha)$ for Cardinals $\alpha$

### Theorem 8.1

1. For all  $d \geq 1$ ,  $h_{2,d}(\aleph_0) = \aleph_0$ .
2. For all  $d \geq 2$ ,  $h_{3,2}(\aleph_0) = \aleph_0$ .
3. For all  $d \geq 2$ ,  $h_{3,3}(\aleph_0) = \aleph_0$ .

### Proof:

- 1) Let  $P$  be a countable subset of  $R^d$ . Define  $COL : \binom{P}{2}$  via  $COL(x, y) = |x - y|$ . By the (standard) infinite canonical Ramsey theorem there is either homog, min-homog, max-homog, or rainbow set of size  $\aleph_0$ . By Lemma 5.5 the set cannot be homog, min-homog or max-homog. Hence it is rainbow.
- 2) This is similar to the proof of part 1, but using the (standard) 3-ary canonical Ramsey theorem and Lemma 7.10.1.
- 3) This is similar to the proof of part 1, but using the (standard) 3-ary canonical Ramsey theorem and Lemma 7.10.2. ■

## 9 Speculation about Higher Dimensions

**Def 9.1** Let  $\Gamma_0(m) = m$  and, for  $a \geq 0$ ,  $\Gamma_{a+1}(m) = 2^{\Gamma_a(m)}$ .

To get lower bounds on  $h_{a,d}(n)$  using our approach you need the following:

1. Upper bounds on  $ER_a(e, k)$ .

- (a) Lefmann and Rödl [23] have an upper bound on  $ER_a(k)$  that involves  $\Gamma_a$ . We are quite confident that this can be modified to obtain an upper bound on  $ER_a(e, k)$  (if  $e \ll k$  which is our case) that involves  $\Gamma_{a-1}$ . We are also quite confident that this can be modified to obtain an upper bound on  $WER_a(e, k)$  that still involves  $\Gamma_{a-1}$  but is better in terms of constants.
  - (b) Shelah [24] has an upper bound on  $ER_a(k)$  that involves  $\Gamma_{a-1}$ . We suspect that this can be modified to obtain an upper bound on  $ER_a(e, k)$  (if  $e \ll k$  which is our case) that involves  $\Gamma_{a-2}$ . We also suspect that this can be modified to obtain an upper bound on  $WER_a(e, k)$  that still involves  $\Gamma_{a-2}$  but is better in terms of constants.
2. The following geometric lemma: There exists a function  $f(a, d)$  such that the following is true: Let  $p_1, \dots, p_n$  be points in  $\mathbb{R}^d$ , no  $a$  points in the same  $(a-2)$ -dimensional space. Color  $\binom{[n]}{a}$  via  $COL(i_1, \dots, i_a) = VOLUME(p_{i_1}, \dots, p_{i_a})$ . If  $I \subset [a]$  then this coloring has no  $I$ -whomog set of size  $f(a, d)$ . We conjecture that this is true.

If our suspicions about  $ER_a(e, k)$  and our conjecture about geometry are correct then the following is true: For all  $a, d$  there is a constant  $\epsilon_{a,d}$  such that

$$(\forall a \geq 3)[h_{a,d}(n) = \Omega((\log^{(a-2)} n)^{\epsilon_{a,d}})].$$

## 10 Open Questions

1. Improve both the upper and lower bounds for  $h_{a,d}$ . A combination of our combinatorial techniques and the geometric techniques of the papers referenced in the introduction may lead to better lower bounds.
2. We obtain  $h_{2,1}(n) = \Omega(n^{1/6}(\log n)^{1/3})$ . The known result,  $h_{2,1}(n) = \Theta(n^{1/2})$ , has a rather difficult proof. It would be of interest to obtain an easier proof of either the known result or a weaker version of it that is stronger than what we have. An easy probabilistic argument yields  $h_{2,1}(n) = \Omega(n^{1/4})$ .
3. Obtain upper bound on  $ER_a(e, k)$ , and geometric lemmas, in order to get nontrivial lower bounds on (1)  $h_{3,d}$  for  $d \geq 4$ , and (2)  $h_{a,d}$  for  $d \geq 4$ , and  $a \geq d$ . See Section 9 for more thoughts on this.
4. Look at a variants of  $h_{a,d}(n)$  with different metrics on  $\mathbb{R}^d$  or in other metric spaces entirely.
5. Look at a variant of  $h_{a,d}(n)$ , which we call  $h'_{a,d}(n)$ , where the only condition on the points is that they are not all on the same  $(a-2)$ -dimensional space. Using the  $n^{1/d} \times \dots \times n^{1/d}$  grid it is easy to show that, for  $a, d \ll n$ ,  $h'_{a,d}(n) \leq O(n^{(a-1)/a})$ .

6. We showed  $h_{3,2}(\aleph_0) = \aleph_0$  and  $h_{3,3}(\aleph_0) = \aleph_0$ . We conjecture that, for  $\aleph_0 \leq \alpha \leq 2^{\aleph_0}$ ,  $h_{a,d}(\alpha) = \alpha$ . This may require a canonical Ramsey theorem where the graph has  $\alpha$  vertices and the coloring function is well behaved.

## 11 Acknowledgments

We would like to thank Tucker Bane, Andrew Lohr, Jared Marx-Kuo, and Jessica Shi for helpful discussions. We would like to thank David Conlon and Jacob Fox for thoughtful discussions, many references and observations, encouragement, and advice on this paper.

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